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Lines of Curvature of a Surface.

By J. Edmund Wright.

It is known that a surface is intrinsically definite if the two fundamental forms ds^2 and ds^2/ρ are given. With the usual notation we represent these by

$$E\,du^2 + 2F\,du\,dv + G\,dv^2$$

and

$$D\,du^2 + 2D'\,du\,dv + D''\,dv^2.$$

In these two forms E, F, G may be given arbitrary values, and then D, D', D'' are subject to three relations. If the parametric lines are lines of curvature F and D' are both zero. In this case D, D'' are subject to three relations. These are not satisfied together unless a certain relation exists among the coefficients E, G of the first fundamental form. Hence if the parametric curves are lines of curvature E and G must be subject to a general condition.

If this condition is satisfied, it appears that D and D'' are in general definite and therefore the surface is intrinsically determinate.

It appears that the condition is equivalent to two differential equations of the fifth order for E and G. In this paper we determine the two equations, and apply the discussion to the particular case where $E = 1$. The remainder of the paper is concerned with a similar consideration in the case of asymptotic lines.

The equations for D, D'' are *

$$\frac{D\,D''}{\sqrt{E}\,G} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) = 0 \tag{1}$$

$$\frac{\partial}{\partial v}\left(\frac{D}{\sqrt{E}}\right) = \frac{D''}{G}\frac{\partial\sqrt{E}}{\partial v}, \tag{2}$$

$$\frac{\partial}{\partial u}\left(\frac{D''}{\sqrt{G}}\right) = \frac{D}{E}\frac{\partial\sqrt{G}}{\partial u}. \tag{3}$$

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* See Bianchi-Lukat, p. 94.

We write $D/\sqrt{E}=X$, $D''/\sqrt{G}=Y$, $E=A^2$, $G=C^2$, $p=\frac{1}{A}\frac{\partial C}{\partial u}$, $q=\frac{1}{C}\frac{\partial A}{\partial v}$, and use suffixes to denote differentiations with respect to u and v. The equations now become

$$XY+p_1+q_2=0, \qquad X_2=qY, \qquad Y_1=pX. \qquad (4)\ (5)\ (6)$$

If we write $p_1+q_2=-P$ we have $\frac{\partial}{\partial v}X^2=2qP$, $\frac{\partial}{\partial u}Y^2=2pP$. Hence $X^2=2\int qP\,dv$, $Y^2=2\int pP\,du$, where an arbitrary function enters through each of the integrals, and

$$P^2=4\int pP\,du\int qP\,dv. \qquad (7)$$

It is clear that, if this relation is satisfied, the equations are compatible, and therefore a surface exists with the given element of length, for which the parametric lines are lines of curvature.

We now write $P^2=a$, $2qP=b$, $2pP=c$, $X^2=\xi$, $Y^2=\eta$ and our equations become $\xi\eta=a$, $\xi_2=b$, $\eta_1=c$.

Consider first the case in which a, b, c, are all different from zero; ξ must satisfy the two equations $\frac{\partial}{\partial u}\left(\frac{a}{\xi}\right)=c$, $\frac{\partial \xi}{\partial v}=b$. Hence

$$\frac{a_1}{c}\xi-\xi^2-\frac{a}{c}\xi_1=0. \qquad (8)$$

If we differentiate this with respect to v we get

$$(a_1 b-ab_1)\,c+(a_{12}c-a_1c_2-2bc^2)\,\xi+(ac_2-a_2c)\,\xi_1=0. \qquad (9)$$

Now if (a_1b-ab_1) and (ac_2-a_2c) are both zero, $a_{12}c-a_1c_2-2bc^2=0$ since $\xi\neq 0$.

If ac_2-a_2c only is zero, (9) determines ξ uniquely and then η is uniquely determined. In any case (8) and (9) determine ξ as the root of a quadratic equation, unless (9) is nugatory, and ξ_1 is determinate. We assume for the present that $ac_2-a_2c\neq 0$. These equations give $\xi=\alpha$, say, and $\xi_1=\beta$. Also $\xi_2=b$. Hence we have two necessary and sufficient conditions for coexistence, namely $\alpha_2=b$, $\alpha_1=\beta$. These two conditions are of the third order in a, b, c, or of the fifth order in A and C. It is obvious that if they are satisfied, values of ξ and η can be determined to satisfy our system of equations, and further, there are only two possible sets of solutions.

We now show that in general there is only one such set; in fact, divide (9) by $ac_2 - a_2 c$ and differentiate again with respect to v. We get

$$\frac{(a_1 b - ab_1)\,(a_{22}c - ac_{22})c}{(ac_2 - a_2 c)^2} + \frac{2a_{12}bc - acb_{12} - 2b^2c^2 + c(a_1 b_2 - 2a_2 b_1)}{ac_2 - a_2 c} + \frac{\partial}{\partial v}\left(\frac{a_{12}c - a_1 c_2 - 2bc^2}{ac_2 - a_2 c}\right)\xi = 0. \tag{10}$$

If the coefficient of ξ in this equation is not zero, ξ is determined uniquely, and hence there is only one set. If this coefficient is zero we form the corresponding set of equations with η instead of ξ, and again we see that ξ and η are uniquely determinate unless

$$\frac{\partial}{\partial u}\left(\frac{a_{12}b - a_2 b_1 - 2b^2c}{ab_1 - a_1 b}\right) = 0.$$

Hence unless both the expressions named are zero the surface having the given linear element, and the parametric lines as lines of curvature, is intrinsically determinate. Looking away from these particular cases we see that the two general equations of condition may be determined by eliminating ξ, ξ_1, from (8), (9), (10), and η, η_2, from the corresponding equations in η. The two equations thus obtained are obviously of the fifth order in A, C, and they are clearly independent, for fifth derivatives enter only through a_{122} in the first, and through a_{112} in the second.

The only case where ξ and η are not determinate arises if (9) is nugatory. We thus have the results:

(i) If the linear element of a surface is given by $ds^2 = A^2 du^2 + C^2 dv^2$, and if the parametric curves are lines of curvature, the surface in general is intrinsically uniquely determinate, and A and C must satisfy two equations involving their fifth derivatives. These two equations of condition are necessary and sufficient.

(ii) In certain cases there may be two surfaces satisfying the given conditions.

(iii) In certain cases there may be an infinite number of such surfaces.

We shall now enquire more closely into the surfaces of classes (ii) and (iii). The equations arising from the identical vanishing of (10) and its correspondent are rather cumbrous, so we consider the question again *ab initio.* We assume, in fact, that the equations $\frac{\partial}{\partial u}\left(\frac{a}{\xi}\right) = c$ and $\frac{\partial \xi}{\partial v} = b$ have two different solutions,

ξ and ξ'. This is equivalent to the case arising from the identical vanishing of (10) and the corresponding equation, with the exception that we must treat independently the case when the quadratic for ξ is a perfect square. We deduce at once that $\xi - \xi' = U$, $a\frac{\xi - \xi'}{\xi\xi'} = V$, when U and V are functions of u and v alone in that order. We still assume that a, b, c are different from zero, and therefore, since $\xi \neq \xi'$, U and V are both different from zero. We now take as new independent variables $\int\sqrt{U}\,du$ and $\int\sqrt{V}\,dv$, and we have $\xi - \xi' = 1$, $\xi\xi' = a$. Hence from the quadratic for ξ we get

$$\frac{a_1 b - ab_1}{a_2 c - ac_2} = 1. \tag{11}$$

$$\left\{\frac{a_{12}c - a_1 c_2 - 2bc^2}{ac_2 - a_2 c} + \frac{a_1}{a}\right\}\frac{a}{c} = \xi + \xi'.$$

Also if we substitute in turn ξ and ξ' in (9) and substract, we deduce

$$a_{12}c - a_1 c_2 - 2bc^2 = 0. \tag{12}$$

Hence $\xi + \xi' = \frac{a_1}{c}$. By substituting the values of the various coefficients in (9) we obtain $\xi_1 = c$, and therefore $c_2 = b_1$. Hence $c = \bar\omega_1$, $b = \bar\omega_2$, where $\bar\omega$ is a function of u and v. Also (11) becomes $a_1\bar\omega_2 - a_2\bar\omega_1 = 0$, and therefore $a = f(\bar\omega)$. With these values the quadratic for ξ is $f'(\bar\omega)\,\xi - \xi^2 - f(\bar\omega) = 0$, and (12) becomes $f'' - 2 = 0$.

$f(\bar\omega)$ must therefore satisfy the two equations $f'' - 2 = 0$ and $1 = (\xi - \xi')^2 = f'^2 - 4f$. The first is seen to be a consequence of the second, and $f = (\bar\omega + \text{const.})^2 - \frac{1}{4}$. Our equations are now all satisfied and if we incorporate the constant into $\bar\omega$ we have

$$a = \bar\omega^2 - \tfrac{1}{4}, \quad b = \bar\omega_2, \quad c = \bar\omega_1;$$

$$\xi = \bar\omega + \tfrac{1}{2}, \quad \xi' = \bar\omega - \tfrac{1}{2}, \quad \eta = \bar\omega - \tfrac{1}{2}, \quad \eta' = \bar\omega + \tfrac{1}{2}.$$

We still have the relations

$$a = (p_1 + q_2)^2, \quad b = -2q(p_1 + q_2), \quad c = -2p(p_1 + q_2).$$

These give $p_1 + q_2 = \sqrt{\bar\omega^2 - \frac{1}{4}}$

$$p = -\frac{\bar\omega_1}{2\sqrt{\bar\omega^2 - \frac{1}{4}}}, \quad q = \frac{-\bar\omega_2}{2\sqrt{\bar\omega^2 - \frac{1}{4}}}$$

Hence

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{\bar\omega^2 - \frac{1}{4}}}\frac{\partial\bar\omega}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{\bar\omega^2 - \frac{1}{4}}}\frac{\partial\bar\omega}{\partial v}\right) + 2\sqrt{\bar\omega^2 - \tfrac{1}{4}} = 0.$$

We write $\bar{\omega} = \frac{1}{2}\cosh\omega$ and this becomes

$$\frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} + \sinh\omega = 0. \tag{13}$$

In this investigation no distinction has been made between real and imaginary quantities, but the connection between surfaces with the same spherical representation as spherical or pseudospherical surfaces is immediate.*

We now consider the case when the quadratic for ξ has equal roots, and the equations of type (10) are nugatory. We have $(a_{12}c - a_1 c_2 - 2bc^2)/(ac_2 - a_2 c)$ equal to a function of u only. Let this function be $\frac{1}{U}\frac{dU}{du}$, and choose $\int \sqrt{U}\, du$ as the variable u. Similarly choose the new variable v. We now have

$$a_{12}c - a_1 c_2 - 2bc^2 = 0,$$
$$a_{12}b - a_2 b_1 - 2b^2 c = 0,$$

and there is still a constant factor at our disposal in each of the variables u, v. From these two equation we deduce

$$\frac{a_1 b}{a_2 c} = \frac{b_1}{c_2},$$

and therefore each of these fractions is equal to

$$\frac{a_1 b - ab_1}{a_2 c - ac_2}.$$

If we substitute these values in (9) we get $\xi_1 = \frac{a_1}{a_2} b$, and therefore $\xi_1 a_2 = \xi_2 a_1$. ξ is therefore a function of a only, say $f(a)$. The quadratic for ξ is

$$\frac{a_1}{c}\xi - \xi^2 = \frac{aa_1 b}{a_2 c},$$

and since this is a perfect square we have

$$a_1 a_2 = 4abc, \quad \xi = \frac{1}{2}\frac{a_1}{c}.$$

Now

$$\xi = f(a), \ \xi_2 = b, \ \frac{\partial}{\partial v}\left(\frac{a}{\xi}\right) = c,$$

and therefore

$$f'(a)a_1 = b, \left(\frac{1}{f} - \frac{af'}{f^2}\right)a_2 = c.$$

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* See Bianchi-Lukat, p. 472. The relation between the two surfaces corresponds to Hazzidaki's transformation. Eisenhart, Amer. Journal, Vol. 27, pp. 113–172, and Amer. Journal, Vol. 28, pp. 47–70.

Hence

$$\left(\frac{f'}{f} - a\frac{f'^2}{f^2}\right)a_1 a_2 = bc,$$

or

$$4\,a\left(\frac{f'}{f} - a\frac{f'^2}{f^2}\right) = 1.$$

We thus have $2af' - f = 0$, or $f = k\sqrt{a}$, where k is a constant. By choosing appropriately the constant factors at our disposal in u and v, we make $k = 1$, and we now have $\xi = \sqrt{a} = \eta$.

Further $\xi_2 = b$, $\eta_1 = c$, and hence

$$b = \frac{\partial\sqrt{a}}{\partial v}. \quad c = \frac{\partial\sqrt{a}}{\partial u}.$$

If we put $a = \bar{\omega}^2$ we have

$$a = \bar{\omega}^2, \; b = \bar{\omega}_2, \; c = \bar{\omega}_1$$
$$\xi = \bar{\omega}, \; \eta = \bar{\omega}.$$

Hence $p_1 + q_2 = \bar{\omega}$, $p = -\frac{1}{2}\frac{\bar{\omega}_1}{\bar{\omega}}$, $q = -\frac{1}{2}\frac{\bar{\omega}_2}{\bar{\omega}}$, and the equation for ω is

$$\frac{\partial}{\partial u}\left(\frac{1}{\bar{\omega}}\,\frac{\partial\bar{\omega}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\bar{\omega}}\,\frac{\partial\bar{\omega}}{\partial v}\right) + 2\,\bar{\omega} = 0,$$

or if we put $-2\,\bar{\omega} = e^{\omega}$

$$\frac{\partial^2\omega}{\partial u^2} + \frac{\partial^2\omega}{\partial v^2} = e^{\omega}. \tag{14}$$

The spherical representation for this class of surfaces is

$$d\sigma^2 = \bar{\omega}\,(du^2 + dv^2).^*$$

We now consider (iii), for which the quadratics for ξ and η are nugatory, and a, b, c are none of them zero. In the first place suppose $ac_2 - a_2 c = 0$, and that the remaining coefficients in (9) do not vanish. We have $\frac{c}{a} = U$, and by taking $\int U du$ as a new variable u, we have $c = a$.

The equations for ξ become

$$\frac{a_1}{a}\xi - \xi^2 = \xi_1.$$

$$(a_{12}\,a - a_1\,a_2 - 2a^2\,b)\,\xi = a\,(ab_1 - a_1\,b),$$

and we have, as before, a unique value of ξ, and two equations of condition between a and b. There is a further limitation since $c = a$. This involves $p_1 + q_2 + 2p = 0$, and therefore

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*In this connection see Bianchi p. 135 sqq.

$$\frac{\partial}{\partial u}\left(e^{2u} p\right) + \frac{\partial}{\partial v}\left(e^{2u} q\right) = 0;$$

hence if ϕ is a certain function of u and v we have

$$e^{2u} p = \frac{\partial \phi}{\partial v}, \quad e^{2u} q = -\frac{\partial \phi}{\partial u},$$

and

$$a = \quad 4e^{-4u}\left(\frac{\partial \phi}{\partial v}\right)^2$$

$$b = -\ 4e^{-4u}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v}.$$

We now suppose that the quadratic for ξ is nugatory. We have $a = c$, and $a_1 b - a b_1 = 0$. Hence by suitable choice of the variable v we may make $a = b$.

In addition, (9) shows that we must have $a a_{12} - a_1 a_2 - 2a^3 = 0$. If this condition is satisfied the equations $\xi\eta = a$, $\xi_2 = a$, $\eta_1 = a$ are compatible, but we must also have

$$-(p_1 + q_2) = 2p = 2q = \sqrt{a}.$$

Hence a must also satisfy

$$\frac{\partial}{\partial u}(\sqrt{a}) + \frac{\partial}{\partial v}(\sqrt{a}) + 2\sqrt{a} = 0.$$

From this equation $\sqrt{a} = e^{-2u} F(u - v)$, where F is an arbitrary function of its argument. But this value of a cannot satisfy the first condition for any value of F. Hence there are no surfaces of the kind sought.

We now consider the possibility hitherto neglected, namely that one of the quantities a, b, c is zero. Going back to the equations (4), (5), (6), we see that the condition requires either p or q to be zero. We assume $q = 0$ and we have

$$XY + p_1 = 0, \quad X_2 = 0, \quad Y_1 = pX,$$

together with $\frac{\partial A}{\partial v} = 0$. Hence A is a function of u only, and we may by suitable choice of u, make it unity. We have now

$$ds^2 = du^2 + C^2 dv^2,$$

and $p = C_1$. The equations are readily solved; there are two possibilities according as X is or is not zero. In the first case the surface is of zero curvature, and $ds^2 = du^2 + (au + b)^2 dv^2$, where a and b are functions of v only.

In the second case let $X = x'$, where x' is the first derivative of a function x of u only. We now have $X = x'$, $C_1 = V_1 \cos x + V_2 \sin x$, $Y = V_1 \sin x - V_2 \cos x$, where V_1 and V_2 are functions of v only.

The conditions C must satisfy, in order that it may be thrown into the given form, are

$$\begin{vmatrix} C_1 & C_{11} & C_{111} \\ C_{12} & C_{112} & C_{1112} \\ C_{122} & C_{1122} & C_{11122} \end{vmatrix} = 0,$$

and

$$\frac{\partial}{\partial u}\left[\frac{C_{11}\, C_{1112} - C_{112}\, C_{111}}{(C_1\, C_{112} - C_{11}\, C_{12})^3}\right] = 0.$$

The spherical representation is given by

$$d\sigma^2 = x'^2\, du^2 + (V_1 \sin x - V_2 \cos x)^2\, dv^2.$$

It readily follows that the surface is that swept out by a curve fixed in a given plane when the plane rolls on a developable surface.*

We now consider the case of a surface with linear element

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2,$$

for which the parametric lines are asymptotic. We have $D = D'' = 0$, and †

$$\frac{\partial}{\partial u}\,\frac{D'}{\sqrt{EG - F^2}} = -2\left\{ {1\ .\ 2 \atop 2} \right\} \frac{D'}{\sqrt{EG - F^2}}$$

$$\frac{\partial}{\partial v}\,\frac{D'}{\sqrt{EG - F^2}} = -2\left\{ {1\ \ 2 \atop 1} \right\} \frac{D'}{\sqrt{EG - F^2}}.$$

Also if K is the total curvature

$$\frac{D'^2}{EG - F^2} = -K.$$

Hence E, F, G must be subject to the two conditions

$$\frac{\partial}{\partial u} \log K = -4\left\{ {1\ \ 2 \atop 2} \right\}$$

$$\frac{\partial}{\partial v} \log K = -4\left\{ {1\ .\ 2 \atop 1} \right\}.$$

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* See also Bianchi-Lukat, p. 166.

† Bianchi-Lukat, p. 92.

These conditions are seen to be equivalent to two, one of the second and the other of the third in derivatives of E, F, G.

As a particular example, consider the case in which the asymptotic lines cut at right angles. The equations of condition give immediately

$$KE^2 = U, \qquad KG^2 = V,$$

where U is a function of u only, V is a function of V only. By appropriate choice of u and v we make $KE^2 = KG^2 = 1$.

We also have the additional equation

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = e^{\frac{1}{2}V},$$

where $K = e^V$.

Hence $$\sqrt{K} = \frac{4f'(u+iv)\,\phi'(u-iv)}{[f(u+iv)+\phi(u-iv)]^2}.$$

Also $H = 0$, and the spherical representation is given by $d\sigma^2 = -\sqrt{K}(du^2 + dv^2)$. We note that since $ds^2 = \frac{1}{\sqrt{K}}(du^2 + dv^2)$ the spherical representation is conformal, and the general properties of minima surfaces may be readily obtained from this point of view.

Suppose next that only one set of parametric curves, say that for which $v =$ constant, is asymptotic. In this case $D = 0$, and

$$\frac{D'}{\sqrt{EG-F^2}} = \sqrt{-K}.$$

In addition we must have

$$\frac{\partial}{\partial u}\sqrt{-K} = -2\left\{ {1\ 2 \atop 2} \right\}\sqrt{-K} + \left\{ {1\ 1 \atop 2} \right\}\frac{D''}{\sqrt{EG-F^2}},$$

$$\frac{\partial}{\partial u}\left(\frac{D''}{\sqrt{EG-F^2}}\right) - \frac{\partial}{\partial v}\sqrt{-K} =$$

$$2\left\{ {1\ 2 \atop 1} \right\}\sqrt{-K} + \left\{ {1\,.\,1 \atop 1} \right\}\frac{D''}{\sqrt{EG-F^2}}.$$

There is thus one equation of condition, of the fourth order, among the quantities E, F, G, and their derivatives, and this is given by substituting the value of D'' given by the first in the second of the above two equations. If this condition is satisfied D' and D'' are determinate and therefore the surface also is intrinsically determinate. As a particular case we readily deduce that if $E=1$, $F=0$, G must equal au^2+bu+c, where a, b, c are three arbitrary functions of v only. This is of course the case of skew surfaces.

The general question as to what conditions are necessary in order that the parametric curves may be conjugates for some surface with a given linear element may also be treated in the same manner as that connected with the lines of curvature. We have to consider the possibility of coexistence of three equations

$$\xi\eta=a, \quad \xi_2=\alpha\xi+\beta, \quad \eta_1=\gamma\eta+\delta,$$

when a, α, β, γ, δ are known functions of u, v, we find that there must exist two relations of the fifth order among E, F, G and their derivatives.

BRYN MAWR, *April* 1906.